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1982 J. Phys. A: Math. Gen. 15 L457

(http://iopscience.iop.org/0305-4470/15/9/004)

J. Phys. A: Math. Gen. **15** (1982) L457–L462. Printed in Great Britain

LETTER TO THE EDITOR

On static, axially symmetric solutions of four-dimensional principal σ models

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Received 10 May 1982

Abstract. We develop an inverse scattering method for generating static, axially symmetric solutions for SU(N) principal σ models.

Recently several papers have appeared discussing different features of the nonlinear σ models in two dimensions (Ogielski *et al* 1978, Tu 1982). However, the higher-dimensional versions of these models also have been used in different branches of theoretical physics. For example the four-dimensional nonlinear chiral model was applied for understanding current algebra results (de Alfaro *et al* 1973), and this model may also play an important role in describing the pion condensate in nuclear physics. On the other hand, the Heisenberg model of the ferromagnet can be considered as an O(3) static nonlinear σ model (Belavin and Polyakov 1975). As was pointed out by Hirayama *et al* (1978) the axially symmetric version of this model is in close connection with the Ernst equation of general relativity describing stationary and axially symmetric vacua.

The self-dual classical solutions of gauge theories can also be obtained from a four-dimensional σ model-like equation (Pohlmeyer 1980, Forgács *et al* 1981a, Horváth and Kiss-Tóth 1981).

Several methods have been worked out to generate solutions of Ernst equations (Harrison 1978, Neugebauer 1979, Belinski and Zakharov 1979) and some of them were even generalised for higher groups (Horváth and Kiss-Tóth 1981). Using these techniques one can construct the asymptotically flat, stationary, axially symmetric vacua in general relativity and, as a completely different application, the static axially symmetric self-dual monopoles for Yang–Mills fields coupled with a Higgs scalar in the adjoint representation (Corrigan and Goddard 1981, Forgács *et al* 1980, 1981b, c).

The aim of the present letter is to show that these methods can be applied successfully for SU(N) principal σ models. We present here how the procedure proposed by Belinski and Zakharov (1979) for the Ernst equations can be reformulated to generate static axially symmetric solutions of SU(N) nonlinear principal σ models. As an application we generate some solutions for the SU(2) case (chiral models).

The usual form of the action for the nonlinear chiral models is

$$S=\frac{1}{2}\int \partial_\mu \Phi_i \partial^\mu \Phi^i \, d^4x \qquad i=0,1,2,3 \tag{1}$$

with the constraint that the length of the field is constant

$$\Phi_i \Phi^i = 1. \tag{2}$$

This action leads to the following field equations

$$\partial_\mu \partial^\mu \Phi^i = \lambda \Phi^i \tag{3}$$

where λ is the Lagrange multiplier field.

A more convenient form of the action is

$$S = \int \partial_\mu g \partial^\mu g^{-1} \, \mathrm{d}^4 x \tag{4}$$

where $g = \Phi_0 + \mathrm{i}\tau_i \Phi_i \in \mathrm{SU}(2)$ and the τ_i are the Pauli matrices. The model is sometimes called principal σ model too. One can readily generalise the model by considering $g \in \mathrm{SU}(N)$ which is the principal $\mathrm{SU}(N)$ σ model.

The corresponding field equations will have the following form

$$\partial^\mu (\partial_\mu g g^{-1}) = 0. \tag{5}$$

In what follows we want to construct classical static, axially symmetric solutions of this model. Then equation (5) reduces to

$$\partial_r (r \partial_r g^{-1}) + \partial_{x_3}(r \partial_{x_3} g^{-1}) = 0 \tag{6}$$

where $r = (x_1^2 + x_2^2)^{1/2}$.

This is a very familiar equation discussed in Belinski and Zakharov's paper (1979) with the only difference that the matrix g is unitary in our case while it was Hermitian in theirs. Therefore the clue for finding solutions of equation (6) is to write up an LA pair defined as

$$D_1 \Psi = \frac{rU_2 - \lambda U_1}{\lambda^2 + r^2} \Psi \qquad D_2 \Psi = \frac{rU_1 + \lambda U_2}{\lambda^2 + r^2} \Psi \tag{7}$$

where we introduced the following notation

$$D_1 = \partial_{x_3} - \frac{2\lambda^2}{\lambda^2 + r^2} \partial_\lambda \qquad D_2 = \partial_r + \frac{2\lambda r}{\lambda^2 + r^2} \partial_\lambda \qquad U_1 = r \partial_r g g^{-1} \qquad U_2 = r \partial_{x_3} g g^{-1} \tag{8}$$

and the complex parameter λ is independent of the variables r and x_3. From the condition of the unitarity of g it follows that the matrices U_i are anti-Hermitian.

The compatibility condition of the system of equations (7) is equation (6). One can see that a natural boundary condition for the matrix $\Psi(\lambda, r, x_3)$ at $\lambda = 0$ is

$$g(r, x_3) = \Psi(0, r, x_3). \tag{9}$$

If we want to generate new solutions of equation (6) it is necessary to know a particular solution of it. Let us assume that this solution is given by g_0, U_{10} and U_{20}; then as a first step we must integrate system (7) to obtain the corresponding solution $\Psi_0(\lambda, r, x_3)$. Then we look for the new solution $\Psi(\lambda, r, x_3)$ in the following form

$$\Psi = \chi \Psi_0. \tag{10}$$

Substituting (10) into (7) we obtain for $\chi(\lambda, r, x_3)$

$$D_1\chi=\frac{rU_2-\lambda U_1}{\lambda^2+r^2}\chi-\chi\frac{rU_{20}-\lambda U_{10}}{\lambda^2+r^2}\qquad D_2\chi=\frac{rU_1+\lambda U_2}{\lambda^2+r^2}\chi-\chi\frac{rU_{10}+\lambda U_{20}}{\lambda^2+r^2}. \tag{11}$$

Taking into account the unitarity of g one can derive that $[\chi^+(\bar{\lambda})]^{-1}$ fulfils the same equation as χ. Therefore, the simplest way to guarantee the unitarity of χ is to impose the condition

$$\chi(\lambda)\chi^+(\bar{\lambda})=I \tag{12}$$

where the bar denotes complex conjugate and + stands for the adjoint.

Furthermore we demand that the matrix χ should tend to unity as $\lambda\to\infty$

$$\chi(\infty)=I.$$

Once we obtain a solution χ of the system (11) we can immediately get a new solution of (6):

$$g=\chi(0)g_0. \tag{13}$$

The soliton solutions of equation (6) correspond to the simple poles of the matrix χ in the complex plane of λ. Therefore, the matrix χ can be written in the following way

$$\chi=I+\sum_{k=1}^{n}\frac{R_k}{\lambda-\mu_k} \tag{14}$$

where n denotes the number of poles. The residues R_k and the pole positions μ_k depend only on the variables r and x_3.

$\chi^{-1}(\lambda)$ is defined in a similar way

$$\chi^{-1}(\lambda)=I+\sum_{k=1}^{n}\frac{S_k}{\lambda-\nu_k}. \tag{15}$$

Taking the residues of the poles μ_k in the relation $\chi\chi^{-1}=I$ one can easily see that

$$R_k\cdot\chi^{-1}(\mu_k)=0 \tag{16}$$

which means that the matrices R_k and $\chi^{-1}(\mu_k)$ are degenerate having the form

$$(R_k)_{ab}=\sum_{i_k=1}^{s_k}n_a^{(k,i_k)}m_b^{(k,i_k)}\qquad [\chi^{-1}(\mu_k)]_{ab}=\sum_{j_k=1}^{N-s_k}q_a^{(k,j_k)}p_b^{(k,j_k)} \tag{17}$$

with $m_b^{(k,i_k)}q_b^{(k,j_k)}=0$ where $1\leqslant s\leqslant N-1$.

From constraint (12) we obtain

$$\mu_k=\bar{\nu}_k\qquad\text{and}\qquad S_k^+=R_k.$$

Substitution of expression (14) into (11) and the supplementary condition (12) completely determine the pole trajectories $\mu_k(r, x_3)$ and the matrices $R_k(r, x_3)$. The requirement that the residues of the second-order poles appearing on the left-hand side of the equation (11) should vanish, determines the form of the function μ_k. These functions must satisfy a pair of differential equations

$$\partial_{x_3}\mu_k+2\mu_k^2(\mu_k^2+r^2)^{-1}=0\qquad \partial_r\mu_k-2r\mu_k(\mu_k^2+r^2)^{-1}=0 \tag{18}$$

whose solutions are roots of

$$\mu_k^2 - 2(\omega_k - x_3)\mu_k - r^2 = 0$$

where ω_k are arbitrary constants (in general complex).

The vectors $m_a^{(k,i_k)}$ defined in (17) can be obtained from equation (11) by requiring that they be satisfied at the poles $\lambda = \mu_k$. They can be constructed from the given solutions $\Psi_0(\lambda, r, x_3)$ as follows

$$m_a^{(k,i_k)} = m_{c_0}^{(k,i_k)}[\Psi_0^{-1}(\mu_k, r, x_3)]_{ca} \tag{19}$$

where the $m_0^{(k,i_k)}$ are s complex linearly independent constant vectors. The remaining task is to obtain $n^{(k,i_k)}$ from condition (12). The final result is that the $n_a^{(k,i_k)}$ are the solutions of a system of $\Sigma_{i=1}^n s_i$ linear equations

$$\sum_{k,\{i_k\}}^{n} n_a^{(k,i_k)}\Gamma^{(k,i_k)(l,j_l)} = -\bar{m}_a^{(l,j_l)} \qquad k, l = 1, 2, \ldots, n \qquad j_l, i_k = 1, 2, \ldots, s_k \tag{20}$$

where Γ is a matrix with elements

$$\Gamma^{(k,i_k)(l,j_l)} = \frac{m_b^{(l,j_l)}\bar{m}_b^{(k,i_k)}}{\bar{\mu}_l - \mu_k}. \tag{21}$$

The form of the new solution g with the aid of (9), (10) and (14) is

$$g = \Psi(0) = \chi(0)\Psi_0(0) = \chi(0)g_0 = \Big(I - \sum_k R_k\mu_k^{-1}\Big)g_0. \tag{22}$$

In order to get a new solution of our equation (6) we must have a unitary g with det $g = 1$. Therefore, we calculate the determinant of g as given in (22), by noticing that the same result can also be obtained iteratively, i.e. applying a similar procedure n times but using $\chi_1(\lambda)$ containing a single pole term only in every step (this means that for $\chi_1(\lambda)$ in equation (14), $n = 1$). Therefore, it is enough to calculate the determinant of χ containing only a single pole. For this case the matrix χ_1 is written as

$$\chi_1 = I + \frac{\mu - \bar{\mu}}{\lambda - \mu}P \tag{23}$$

where P is a Hermitian projector

$$P^2 = P \qquad P^+ = P \qquad \det P = 0.$$

The determinant of (23) at $\lambda = 0$ is

$$\det \chi_1(0) = (\bar{\mu}/\mu)^s$$

where s is the dimension of the image subspace of P.

It is simple to prove that the renormalised g^{ph}

$$g^{\mathrm{ph}} = \prod_i \left(\frac{\mu_i}{\bar{\mu}_i}\right)^{s_i/N} \chi(0)g_0 \tag{24}$$

also solves equation (6) giving a physical solution $g^{\mathrm{ph}} \in \mathrm{SU}(N)$.

As an example we show here how one can calculate the single-soliton solutions. Let us suppose for simplicity that $\omega = \mathrm{i}\alpha$, which means that the soliton is positioned at $x_3 = 0$. Then it is useful to write the final result in oblate spheroidal coordinates defined as

$$r=\alpha[(1+\sigma^2)(1-\tau^2)]^{1/2} \qquad x_3=\alpha\sigma\tau. \tag{25}$$

In these coordinates μ takes the form

$$\mu=\alpha(1-\tau)(\mathrm{i}+\sigma). \tag{26}$$

Using (23) and (26) the SU(N) one-soliton solution is

$$g^{\mathrm{ph}}=\left(\frac{\sigma+\mathrm{i}}{\sigma-\mathrm{i}}\right)^{s/N}\left(g_0-\frac{2\mathrm{i}}{\sigma+\mathrm{i}}Pg_0\right). \tag{27}$$

In order to specify the projector P one must choose a special g_0.

Looking for SU(2) solutions one may take the following seed solution

$$g_0=\begin{pmatrix} G & 0 \\ 0 & \bar{G}\end{pmatrix} \qquad G=r^{\mathrm{i}\kappa}\exp[\mathrm{i}(mx_3+\delta)] \tag{28}$$

where κ, m and δ are arbitrary real constants. The corresponding solution of the linear system (7) is

$$\Psi_0(\lambda)=\begin{pmatrix} T(\lambda) & 0 \\ 0 & \bar{T}(\lambda)\end{pmatrix} \tag{29}$$

where

$$T=(r^2-2\lambda x_3-\lambda^2)^{\mathrm{i}\kappa/2}\exp\{\mathrm{i}[m(x_3+\tfrac{1}{2}\lambda)+\delta]\}.$$

Therefore, the projector P appearing in (27) can be expressed as

$$\begin{aligned} &P_{ab}=\bar{m}_a m_b \\ &m=[m_{01}\bar{T}(\mu),\, m_{02}T(\mu)] \qquad m_{01}^2+m_{02}^2=1 \\ &T(\mu)=[-2\mathrm{i}\alpha(1-\tau)(\mathrm{i}+\sigma)]^{\mathrm{i}\kappa/2}\exp \mathrm{i}\{m[\sigma\tau+\tfrac{1}{2}(1-\tau)(\mathrm{i}-\sigma)]+\delta\}\end{aligned} \tag{30}$$

and the resulting g^{ph} is

$$g_{ab}^{\mathrm{ph}}=(1+\sigma^2)^{-1/2}[(\sigma+\mathrm{i})\delta_{ac}-2\mathrm{i}\bar{m}_a m_c]g_{0cb}. \tag{31}$$

The discussion of the so-called reduction problem needs further study. For example, if we want to obtain solutions for the Heisenberg model, we must impose the condition: $g^+=-g$ as well. This would restrict the form of (14), namely, only an even number of poles could appear in it.

We should like to thank Drs P Forgács and L Palla for discussions and for reading the manuscript.

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